

Absolutely homotopy-cartesian squares

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ABSTRACT. We call a diagram \mathcal{D} absolutely cartesian if $F(\mathcal{D})$ is homotopy cartesian for all homotopy functors F . This is a sensible notion for diagrams in categories \mathcal{C} where Goodwillie's calculus of functors may be set up for functors with domain \mathcal{C} . We prove a classification theorem for absolutely cartesian squares of spaces and state a conjecture of the classification for higher dimensional cubes.

Let I be a small indexing category with initial object \emptyset and final object 1 . A diagram \mathcal{D} in a category \mathcal{C} is a functor $I \rightarrow \mathcal{C}$; we restrict ourselves here to \mathcal{C} being spaces. This diagram is cartesian¹ when $\mathcal{D}(\emptyset)$ is equivalent to the homotopy limit of \mathcal{D} over I with \emptyset removed, denoted $\operatorname{holim}_{I_0} \mathcal{D}$ or $\operatorname{holim}_{\emptyset} \mathcal{D}$ when I is clear from context. Similarly, \mathcal{D} is cocartesian if $\mathcal{D}(1)$ is equivalent to the homotopy colimit over I with the final object removed, denoted $\operatorname{hocolim}_I \mathcal{D}$; as in the cartesian case, the I subscript is omitted if clear from context and we write $\operatorname{hocolim}_1$. A functor F is a homotopy functor if it is weak-equivalence-preserving. We call a diagram \mathcal{D} absolutely (co)cartesian if $F(\mathcal{D})$ is homotopy (co)cartesian for all homotopy functors F . Note that a diagram is an $(n+1)$ cube if it is indexed by $I = \mathcal{P}([n])$, the powerset on $[n] = \{0, 1, \dots, n\}$.

1. Statements of Results and Conjectures

We prove the following classification theorem for absolutely cartesian squares:

THEOREM 1.1. *A square of spaces is absolutely cartesian if and only if it is a map of two absolutely cartesian 1-cubes. That is, of the following form (the other two maps may also be equivalences):*

$$\begin{array}{ccc} A & \xrightarrow{\sim} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{\sim} & D \end{array}$$

Theorem 1.1 is the base case of our following conjecture:

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¹In keeping with conventions of Goodwillie's calculus of functors, we only deal with homotopy cartesian diagrams, so omit the "homotopy" modifier.

CONJECTURE 1.2. *An n -cube of spaces is absolutely cartesian if and only if it can be written as either a map of two absolutely cartesian $(n-1)$ -cubes or a chain of compositions of n -cubes of these types.*²

It should be clear that building up an n -cube inductively as maps of these absolutely cartesian squares and compositions of such cubes will yield an absolutely cartesian n -cube, which is the \Leftarrow direction of the if and only if. To be clear, two cubes \mathcal{C}, \mathcal{D} may be composed if they can be written $\mathcal{C} : X \rightarrow Y$ and $\mathcal{D} : Y \rightarrow Z$; their composition is then $\mathcal{C} \circ \mathcal{D} : X \rightarrow Z$. Geometrically, this looks like “glueing” the cubes along their shared face. We give an example in the next section. By chain of compositions, we mean compositions of possibly more than two cubes, e.g. $\mathcal{C} \circ \mathcal{D} \circ \mathcal{E}$ where $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are all n -cubes built inductively up from maps of absolutely cartesian squares.

It is not yet certain if the other direction is true. We may observe that the absolutely cartesian squares are also absolutely cocartesian. Thus, we make an additional conjecture that

CONJECTURE 1.3. *An n -cube is absolutely cartesian if and only if it is absolutely cocartesian.*

If we include contravariant functors, we can show this conjecture for $n = 2$, and we will comment on this after the proof for cartesian squares, which is in the following section.

We will present partial results towards Conjecture 1.3 in section 3; this includes a positive verification of the conjecture when restricting to functors which land in 1-connected spaces (including the identity implies that the spaces in the diagram must originally be 1-connected as well).

The section after that is about a family of 3-cubes which are absolutely cocartesian and cartesian and which are not expressible as a map of two absolute cartesian squares, but as a composition of 3-cubes of that form. We end with applications and related work.

2. Proof of Classification for Squares

PROOF OF THEOREM 1.1.³ This relies on switching briefly to the setting of spectra and using this to deduce properties of the original diagram of spaces. We also point out that it suffices to prove that either $B \rightarrow D$ or $C \rightarrow D$ is an equivalence, since equivalences are stable under homotopy pullback. That is, it implies that the mirroring map, $A \rightarrow C$ or $A \rightarrow B$, is also an equivalence.

Consider an absolutely cartesian square of spaces:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

²There should be some way to express this as the cubes being “generated by” those built out of absolutely cartesian squares.

³The current form (and brevity) of this proof is influenced heavily by conversations between the author and Tom Goodwillie about developing a clearer route towards attacking the more general conjecture.

Now apply the functor $\Sigma^\infty \text{Map}(D, -)$ to our square:

$$\begin{array}{ccc} \Sigma^\infty \text{Map}(D, A) & \longrightarrow & \Sigma^\infty \text{Map}(D, B) \\ \downarrow & & \downarrow \\ \Sigma^\infty \text{Map}(D, C) & \longrightarrow & \Sigma^\infty \text{Map}(D, D) \end{array}$$

By assumption, this resultant square is still cartesian. Since the square is in spectra, we know that it is also cocartesian. Recall that Σ^∞ commutes with colimits.

We then have the following chain of equivalences:

$$\begin{array}{ccc} \pi_0 \Sigma^\infty \text{hocolim}(\text{Map}(D, B) \leftarrow \text{Map}(D, A) \rightarrow \text{Map}(D, C)) & \simeq & \pi_0 \Sigma^\infty \text{Map}(D, D). \\ \parallel & & \parallel \\ H_0(\text{hocolim}(\text{Map}(D, B) \leftarrow \text{Map}(D, A) \rightarrow \text{Map}(D, C))) & & H_0(\Sigma^\infty \text{Map}(D, D)) \\ \parallel & & \parallel \\ \mathbb{Z}[\pi_0(\text{hocolim}(\text{Map}(D, B) \leftarrow \text{Map}(D, A) \rightarrow \text{Map}(D, C)))] & & \mathbb{Z}[\pi_0 \text{Map}(D, D)] \end{array}$$

We can interpret this as telling us that $(\pi_0 \text{Map}(D, B) \cup \pi_0 \text{Map}(D, C)) / \sim$ surjects onto $\pi_0 \text{Map}(D, D)$. Consider $id \in \text{Map}(D, D)$. This then has a preimage (up to homotopy) in $\text{Map}(D, B)$ and/or $\text{Map}(D, C)$; assume $\text{Map}(D, B)$. This gives a section $D \rightarrow B$. We can then rewrite our original diagram with our new map in the pre-image of the identity. This is Figure 1.

$$\begin{array}{ccc} & D & \\ & \downarrow & \\ A & \longrightarrow & B \xrightarrow{id} D \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

FIGURE 1. New information included in diagram

We can add the homotopy pullback of $(A \rightarrow B \leftarrow D)$ to the diagram. Then the whole diagram is a pullback, being a composition of pullback squares. This lets us pull back the identity map, as in Figure 2.

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ id \swarrow & A \longrightarrow B & \searrow id \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

FIGURE 2. Adding the pullback of the top punctured square and pulling back the identity

The whole diagram is itself absolutely cartesian (having two facing maps which are equivalences). Since the bottom and entire squares are both absolutely cartesian, so is the top square, shown again in Figure 3.

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

FIGURE 3. “top” square

Now that the top square is known to be absolutely cartesian, we can proceed in the same as we did with the original square, and obtain a section from B to D or A . If the section is to D , we are done, as we already have a splitting from D to B and having another the other direction gives us an equivalence between B and D .

Otherwise, we work in the other direction. We add our section $B \rightarrow A$ to our diagram, in Figure 4, shown without the other equivalences.

$$\begin{array}{ccccc} & C & \longrightarrow & D & \\ & \downarrow & & \downarrow & \\ B & \longrightarrow & A & \longrightarrow & B \\ & \downarrow & & \downarrow & \\ & C & \longrightarrow & D & \end{array}$$

FIGURE 4. Adding the section $B \rightarrow A$

Then we pull back the upper left square. The square comprised of the upper left and right squares together is then a cartesian square, with bottom map an equivalence. These are stable under pullback, meaning that the identity map $B \rightarrow B$ is pulled back, this time to the “top”. Thus we know the pullback of the left square is equivalent to D , so the top two squares are as in Figure 5. This implies that the left square is also absolutely cartesian, as the entire and the right ones are.

$$\begin{array}{ccccc} D & \xrightarrow{id} & C & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{id} & A & \longrightarrow & B \end{array}$$

FIGURE 5. Pulled back identity to top of diagram

Then we return to A , and the (now) absolutely cartesian square in the left of Figure 5. In the same way as before, we get a section $A \rightarrow C$ or $A \rightarrow B$. If $A \rightarrow B$ is

a section, we are done – the one-sided inverse (our original section) has an inverse on the other side and $A \xrightarrow{\sim} B$, which implies immediately that $D \simeq C$ since it occurs in the cartesian diagram on the left side of Figure 5 .

If $A \rightarrow C$ instead is a section, we are done. This is because the map $C \rightarrow A$ was a section obtained earlier. We conclude $A \xrightarrow{\sim} C$, which implies immediately that $D \simeq B$ since it occurs in the cartesian diagram on the left side of Figure 5 and equivalences are pulled back.

□

REMARK 2.1. For absolutely **cocartesian** squares, if we allow our homotopy functors to also be possibly contravariant, then we can establish that they are of the same form as absolutely cartesian squares. The proof is parallel to that for cartesian squares, with the functor $\Sigma^\infty \text{Map}(D, -)$ replaced by $\Sigma^\infty \text{Map}(-, A)$.

3. Partial results for Conjecture 1.3

As observed by the anonymous reviewer, the following weakened form of the conjecture already holds:

PROPOSITION 3.1. *If one restricts to n -cubes of 1-connected spaces and homotopy functors which take values in 1-connected-spaces, Conjecture 1.3 holds. The direction (absolutely cocartesian implies absolutely cartesian) holds in the weaker condition of nilpotent⁴ spaces and functors taking values in nilpotent spaces.*

PROOF. This is also following the reviewer.

- (1) **Functors with nilpotent target, abs cocartesian \Rightarrow abs cartesian.** Let X be absolutely cocartesian. The functor Σ^∞ from Spaces to Spectra preserves the cocartesianness, and in Spectra, diagrams are cocartesian iff cartesian. Ω^∞ from Spectra to Spaces preserves cartesianness, so $QX := \Omega^\infty \Sigma^\infty X$ is cartesian, in addition to remaining cocartesian. Repeated applications of Q will clearly retain this property; that is, $QQ \cdots QX$ will be cartesian and cocartesian.

As Ω^∞ and Σ^∞ are an adjoint pair and Q the associated monad⁵, there is an associated cosimplicial “ Q -completion” (a.k.a \mathbb{Z} -nilpotent completion) for any space. For a space X , the Q -completion of X is the homotopy limit of the cosimplicial space which arises naturally from iterating the monadic maps $X \rightarrow Q(X)$ and $QQX \rightarrow QX$.

$$QX \rightleftarrows QQX \rightleftarrows \cdots$$

The same line of reasoning holds with X replaced by $F(X)$ (since $F(X)$ is also absolutely cocartesian), so X is absolutely cartesian.

- (2) **Functors with 1-connected target, absolutely cartesian \Rightarrow absolutely cocartesian.** Let X be absolutely cartesian. Then $F(X)$ and $\Sigma^\infty F(X)$ for all hofunctors $F : \text{Top} \rightarrow \text{Top}$ are also cartesian; in particular, $\Sigma^\infty F(X)$ is also cocartesian. Since F takes values in 1-connected spaces, this is sufficient to conclude that $F(X)$ itself is cocartesian. This is for all hofunctors F , so X is absolutely cocartesian.

⁴A space X is nilpotent when $\pi_1(X)$ is a nilpotent group. 1-connected spaces are trivially nilpotent.

⁵Also referred to as a “triple”.

□

It was also pointed out by Goodwillie in a discussion with the author that

PROPOSITION 3.2. *If (absolutely cocartesian \Rightarrow absolutely cartesian), then (absolutely cartesian \Rightarrow absolutely cocartesian).*

PROOF SKETCH: Let X be absolutely cartesian. Then for all F, G hofunctors and A, B in the appropriate categories, $\text{Map}(F(\text{Map}(G(X), A)), B)$ is also cartesian. Unwrapping the dependencies and keeping in mind that $\text{Map}(-, Y)$ takes cocartesian to cartesian, we get that $\text{Map}(G(X), A)$ is absolutely cocartesian. Apply our hypothesis and that $\text{Map}(-, Y)$ takes cocartesian to cartesian to conclude that X is also absolutely cocartesian. □

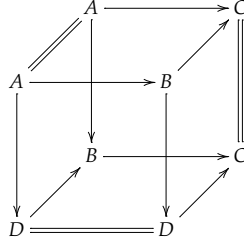
4. An Absolutely Cocartesian and Cartesian 3-cube

The original form of Conjecture 1.2 was as follows:

An $(n+1)$ -cube of spaces X is absolutely cartesian iff there are absolutely cartesian n -cubes Y, Z such that $X : Y \rightarrow Z$.

This was corrected to the current form of the conjecture due to the following illustrative example; a cube which may be expressed as the *composition* of two cubes of the aforementioned type without being a map of two such absolutely (co)cartesian squares.

Given maps $A \rightarrow B \rightarrow D \rightarrow B \rightarrow C$ with the condition that $B \rightarrow D \rightarrow B$ is equivalent to the identity, the following 3-cube may be assembled: ⁶:



Now that we have more complicated diagrams, we have chosen to denote equivalences by equality so that it is clear which maps are equivalences.

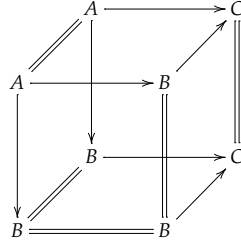
It is possible to first establish absolute cartesianness and cocartesianness independent of the decomposition, but this is superfluous once we have the decomposition. We will then just provide the decomposition.

4.1. Factorization. Despite not being a map of two absolutely (co)cartesian squares, the 3-cube ⁷ may be expressed as a composition of two 3-cubes which *are* of that form. This relies on the ability to express B as a retract of D . We compose

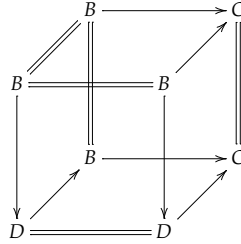
⁶We thank the referee for this example, which made it clear that we need to include not just *maps* of $(n-1)$ cubes.

⁷This factorization related to one pointed out by Tom Goodwillie.

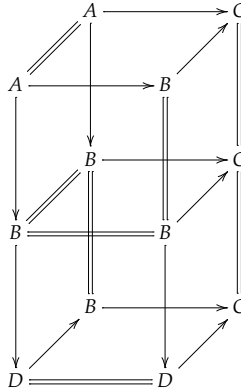
the cube



with



to get our original 3 cube as the total cube of the composition ('glueing' the first cube atop the second):



5. Applications and Related Work

We end with a few remarks on extending this and other approaches. Paré[2] studies strict colimits which are preserved by all functors, and calls such colimits *absolute*, a naming convention which we have chosen to follow by calling our *homotopy* diagrams *absolute* when preserved by all homotopy functors. Street works in an enriched setting and states his results in terms of distributors[3]. It is not clear at the moment how applicable their results are in this setting. The first step would be to switch to considering simplicial functors, which are (roughly) as good as homotopy functors, to work enriched.

The original goal to classifying absolutely cartesian cubes was to get “wrong way” maps, from holims of cubes of one dimension to ones of a higher dimension, in a certain diagram related to the E_1 page of the spectral sequence associated to a cosimplicial space. These are going the wrong way inasmuch as natural maps between diagrams are usually from lower to higher dimension, which induces a

map from the holim of the higher dimensional diagram to the holim of the lower dimensional diagram.

A map of cubes *of the same dimension*, $A \rightarrow B$, induces maps on the homotopy limits of the cubes $\operatorname{holim} A \rightarrow \operatorname{holim} B$ (also for the punctured homotopy limits, $\operatorname{holim}_\emptyset$). If A and B are diagrams, with B cartesian, $\operatorname{holim}_\emptyset(A \rightarrow B) \simeq \operatorname{holim}_\emptyset A$. That is, a way to take an n cube and produce an $(n+1)$ cube with equivalent (punctured) homotopy limit is to find a cartesian n cube to which it maps naturally. We would also like do these constructions only once for all homotopy functors, so the cube we are mapping to not only needs to be cartesian, but with cartesianness preserved by all homotopy functors.

References

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3. R. Street, Absolute colimits in enriched categories, Cahiers de Top. et Geom. Di **24** (1983), no. 4, 377–379.

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